SOLUTIONS TO SELECTED QUESTIONS IN HOMEWORK 5

MATH 241

18.2.7

Proof. $\tan z$ is analytic outside the zeroes of $\cos z$, i.e., $\pm\frac{\pi}{2} + 2n\pi$. But all these points are out of the unit circle, so $\oint_C \tan z dz = 0$. □

18.2.19

Proof. Partition of fraction:

$$\frac{z-1}{z(z-i)(z-3i)} = \frac{A}{z} + \frac{B}{z-i} + \frac{C}{z-3i}$$

$$z - 1 = A(z-i)(z-3i) + Bz(z-3i) + Cz(z-i)$$

Compare the coefficients we get

$$A + B + C = 0$$

$$-4iA - 3iB - iC = 1$$

$$-3A = -1$$

Solve out

$$A = \frac{1}{3}, B = -\frac{1}{2} + \frac{1}{2}i, C = \frac{1}{6} - \frac{1}{2}i$$

So

$$\oint_C \frac{z-1}{z(z-i)(z-3i)} = \frac{1}{3}\oint_C \frac{1}{z}dz + (-\frac{1}{2} + \frac{1}{2}i)\oint_C \frac{1}{z-i}dz + (\frac{1}{6} - \frac{1}{2}i)\oint_C \frac{1}{z-3i}dz$$

The contour C only contains i among the three points $0, i, 3i$ where the function is not analytic, so the answer is $(-\frac{1}{2} + \frac{1}{2}i) \cdot 2\pi i = -\pi - \pi i$. □

18.3.1

Proof. Method (1): Since $4z - 1$ is analytic on the entire complex plane, we can change the path to the straight segment from $-i$ to i. The path is parametrized as $z(t) = it$, t from -1 to 1. So

$$\int_C (4z-1)dz = \int_{-1}^{1} (4it-1)idt = \int_{-1}^{1} (-4t)dt - i\int_{-1}^{1} dt = -2i$$

Method (2): An antiderivative of $4z - 1$ is $2z^2 - z$ on the complex plane, so

$$\int_C (4z-1)dz = 2z^2 - z|_{-i}^{i} = -2i$$

□

18.3.23

Proof. To find an antiderivative of ze^z, one way is to pretend it was a real function and integrate it as a real function, after you get your guess, view your guess as a complex function and differentiate it as a complex function, see whether it gives back your function ze^z. So first calculate the indefinite integral of the real function xe^x.

$$\int xe^x dx = \int xd(e^x) = xe^x - \int e^x dx = xe^x - e^x + C$$

So $ze^z - e^z$ is one of your candidate for the antiderivative of ze^z. We check by differentiation:

$$\frac{d}{dz}(ze^z - e^z) = e^z + ze^z - e^z = ze^z$$

So $ze^z - e^z$ is indeed an antiderivative.

Therefore

$$\int_i^{1+i} ze^z dz = ze^z - e^z|_i^{1+i} = (1+i)e^{1+i} - e^{1+i} - ie^i + e^i = ie(\cos 1 + i\sin 1) - i(\cos 1 + i\sin 1) + (\cos 1 + i\sin 1)$$

$$= (-e\sin 1 + \sin 1 + \cos 1) + i(e\cos 1 - \cos 1 + \sin 1)$$

□

Fall 09, #3:

Proof. An antiderivative of $z\cos(z^2)$ is $\frac{1}{2}\sin(z^2)$. So

$$\int_0^{-1-i} z\cos(z^2)dz = \frac{1}{2}\sin(z^2)|0^{-1-i} = \frac{1}{2}(\sin(-1-i)^2) = \frac{1}{2}\sin(-2i) = \frac{1}{2}\frac{e^{i\cdot(-2i)} - e^{-i\cdot(-2i)}}{2i} = \frac{-e^2 + e^{-2}}{2}i$$

□

Spring 09, #5:

Proof. The most tedious part is to calculate the partition of fraction of $\frac{1}{z^4+10z^2+9}$. The denominator factorizes as $z^4 + 10z^2 + 9 = (z^2+1)(z^2+9) = (z-i)(z+i)(z-3i)(z+3i)$. Establish equation

$$\frac{1}{z^4 + 10z^2 + 9} = \frac{A}{z-i} + \frac{B}{z+i} + \frac{C}{z-3i} + \frac{D}{z+3i}$$

Get

$$1 = A(z+i)(z^2+9) + B(z-i)(z^2+9) + C(z-3i)(z^2+1) + D(z+3i)(z^2+1)$$

Compare and get

$$A + B + C + D = 0$$

$$iA - iB + 3iC - 3iD = 0$$

$$9A + 9B + C + D = 0$$

$$9iA - 9iB + 3iC - 3iD = 1$$

From the first and third equations you can solve $A+B = 0, C+D = 0$. From the second and fourth equations you can solve $A - B = -\frac{i}{8}, C - D = \frac{i}{24}$. So eventually we get

$$A = -\frac{i}{16}, B = \frac{i}{16}, C = \frac{i}{48}, D = -\frac{i}{48}$$

Therefore the contour integral simplifies to

$$-\frac{i}{2\pi}\oint_C \frac{1}{z-i}dz + \frac{i}{2\pi}\oint_C \frac{1}{z+i}dz + \frac{i}{6\pi}\oint_C \frac{1}{z-3i}dz - \frac{i}{6\pi}\oint_C \frac{1}{z+3i}dz$$

(a) C only encloses $3i$, so the answer is $\frac{i}{6\pi}\cdot 2\pi i = -\frac{1}{3}$; (b) C encloses $-i, i, -3i$, so the answer is $(-\frac{i}{2\pi} + \frac{i}{2\pi} - \frac{i}{6\pi})\cdot 2\pi i = \frac{1}{3}$; (c) C encloses $-i, -3i$, so the answer is $(\frac{i}{2\pi} - \frac{i}{6\pi})\cdot 2\pi i = -\frac{2}{3}$; (d) C only encloses i, so the answer is $-\frac{i}{2\pi}\cdot 2\pi i = 1$; (e) C does not enclose anyone among $\pm i, \pm 3i$, so the answer is 0. □